
Denotational semantics

April 7, 2009

Overview

▷ Overview

Direct style semantics:
specification

Fixed point theory

Direct style semantics:
existence

An equivalence result

- Direct style semantics: specification
- Fixed point theory
- Direct style semantics: existence
- An equivalence result

Overview

Direct style semantics:
▷ specification

Idea

Semantic functions so far

DS for While

S_{ds} for composition

S_{ds} for condition

S_{ds} for while

Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on the fixed point

Case 1: terminates

Case 2: loops locally

Case 3: loops globally

Example 5.1: fixed points

Generalizing experience

Fixed point theory

Direct style semantics:
existence

Direct style semantics: specification

Idea

Overview

Direct style
semantics:
specification

▷ Idea
Semantic
functions so far
DS for While
 S_{ds} for
composition
 S_{ds} for condition
 S_{ds} for while
Example 5.1
Problem
Example 5.2
Example 5.3
Requirements on
the fixed point
Case 1: terminates
Case 2: loops
locally
Case 3: loops
globally
Example 5.1: fixed
points
Generalizing
experience

Fixed point theory

Direct style
semantics:
existence

In *denotational semantics* we are interested

- in *effects*
- not *how*

a function is executed.

The hallmark

- semantic functions - *compositional*
 - semantic clause for each of basis elements
 - semantic clause for each of compositional elements
 - ▷ defined through immediate constituents

Semantic functions so far

Overview

Direct style semantics: specification

Idea

Semantic functions so far

DS for While

\mathcal{S}_{ds} for composition

\mathcal{S}_{ds} for condition

\mathcal{S}_{ds} for while

Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on the fixed point

Case 1: terminates

Case 2: loops locally

Case 3: loops globally

Example 5.1: fixed points

Generalizing experience

Fixed point theory

Direct style semantics: existence

- $\mathcal{A} : \mathbf{Aexp} \rightarrow (\mathbf{State} \mapsto \mathbf{Z})$ and $\mathcal{B} : \mathbf{Bexp} \rightarrow (\mathbf{State} \mapsto \mathbf{T})$ are ok
- $\mathcal{S}_{ns}, \mathcal{S}_{sos} : \mathbf{Stm} \rightarrow (\mathbf{State} \hookrightarrow \mathbf{State})$
 - not defined compositionally
 - ▷ $[\mathbf{while}_{sos}]$ defined by
$$\langle \mathbf{while } b \mathbf{ do } S, s \rangle \Rightarrow \langle \mathbf{if } b \mathbf{ then } (S; \mathbf{while } b \mathbf{ do } S) \mathbf{ else } skip, s \rangle$$
- $\mathcal{S}_{ds} : \mathbf{Stm} \rightarrow (\mathbf{State} \hookrightarrow \mathbf{State})$

DS for While

Overview

Direct style
semantics:
specification

Idea
Semantic
functions so far
▷ DS for While
 \mathcal{S}_{ds} for
composition

\mathcal{S}_{ds} for condition

\mathcal{S}_{ds} for while

Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on
the fixed point

Case 1: terminates

Case 2: loops
locally

Case 3: loops
globally

Example 5.1: fixed
points

Generalizing
experience

Fixed point theory

Direct style
semantics:
existence

$$\mathcal{S}_{ds} \llbracket x := a \rrbracket s = s[x \mapsto \mathcal{A} \llbracket a \rrbracket s]$$

$$\mathcal{S}_{ds} \llbracket \text{skip} \rrbracket = \text{id}$$

$$\mathcal{S}_{ds} \llbracket S_1; S_2 \rrbracket = \mathcal{S}_{ds} \llbracket S_2 \rrbracket \circ \mathcal{S}_{ds} \llbracket S_1 \rrbracket$$

$$\mathcal{S}_{ds} \llbracket \text{if } b \text{ then } S_1 \text{ else } S_2 \rrbracket = \text{cond}(\mathcal{B} \llbracket b \rrbracket, \mathcal{S}_{ds} \llbracket S_1 \rrbracket, \mathcal{S}_{ds} \llbracket S_2 \rrbracket)$$

$$\mathcal{S}_{ds} \llbracket \text{while } b \text{ do } S \rrbracket = \text{FIX } F$$

$$\text{where } F g = \text{cond}(\mathcal{B} \llbracket b \rrbracket, g \circ \mathcal{S}_{ds} \llbracket S \rrbracket, \text{id})$$

- id is the identity function
- Auxiliary functions FIX and cond

\mathcal{S}_{ds} for composition

$$\begin{aligned} \mathcal{S}_{ds}[[S_1; S_2]] s &= \mathcal{S}_{ds}[[S_2]] \circ \mathcal{S}_{ds}[[S_1]] s \\ &= \begin{cases} s'' & \text{if there exists } s' \text{ such that } \mathcal{S}_{ds}[[S_1]] s = s' \\ & \text{and } \mathcal{S}_{ds}[[S_2]] s' = s'' \\ \underline{\text{undef}} & \text{if } \mathcal{S}_{ds}[[S_1]] s = \underline{\text{undef}} \\ & \text{or there exists } s' \text{ such that } \mathcal{S}_{ds}[[S_1]] s = s' \\ & \text{but } \mathcal{S}_{ds}[[S_2]] s' = \underline{\text{undef}} \end{cases} \end{aligned}$$

\mathcal{S}_{ds} for condition

cond :

$(\mathbf{State} \rightarrow \mathbf{T}) \times (\mathbf{State} \hookrightarrow \mathbf{State}) \times (\mathbf{State} \hookrightarrow \mathbf{State}) \rightarrow (\mathbf{State} \hookrightarrow \mathbf{State})$

$$\text{cond}(p, g_1, g_2) = \begin{cases} g_1 s & \text{if } p s = \mathbf{tt} \\ g_2 s & \text{if } p s = \mathbf{ff} \end{cases}$$

$$\begin{aligned} \mathcal{S}_{ds}[\text{if } b \text{ then } S_1 \text{ else } S_2]s &= \text{cond}(\mathcal{B}[b], \mathcal{S}_{ds}[S_1], \mathcal{S}_{ds}[S_2])s \\ &= \begin{cases} s' & \text{if } \mathcal{B}[b]s = \mathbf{tt} \text{ and } \mathcal{S}_{ds}[S_1]s = s' \\ & \text{or if } \mathcal{B}[b]s = \mathbf{ff} \text{ and } \mathcal{S}_{ds}[S_2]s = s' \\ \underline{\text{undef}} & \text{if } \mathcal{B}[b]s = \mathbf{tt} \text{ and } \mathcal{S}_{ds}[S_1]s = \underline{\text{undef}} \\ & \text{or if } \mathcal{B}[b]s = \mathbf{ff} \text{ and } \mathcal{S}_{ds}[S_2]s = \underline{\text{undef}} \end{cases} \end{aligned}$$

\mathcal{S}_{ds} for while

Overview

Direct style
semantics:
specification

Idea

Semantic
functions so far

DS for While

\mathcal{S}_{ds} for
composition

\mathcal{S}_{ds} for condition

▷ \mathcal{S}_{ds} for while

Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on
the fixed point

Case 1: terminates

Case 2: loops
locally

Case 3: loops
globally

Example 5.1: fixed
points

Generalizing
experience

Fixed point theory

Direct style
semantics:
existence

- the effect must be equal
if b then $(S; \text{while } b \text{ do } S)$ else skip
- which gives
$$\mathcal{S}_{ds}[\text{while } b \text{ do } S] = \text{cond}(\mathcal{B}[b], \mathcal{S}_{ds}[\text{while } b \text{ do } S] \circ \mathcal{S}_{ds}[S], \text{id})$$
- not compositional

However,

- $\mathcal{S}_{ds}[\text{while } b \text{ do } S]$ is a *fixed point* of
- $F g = \text{cond}(\mathcal{B}[b], g \circ \mathcal{S}_{ds}[S], \text{id})$

Example 5.1

Overview

Direct style semantics: specification

Idea

Semantic functions so far
DS for While
 S_{ds} for composition

S_{ds} for condition

S_{ds} for while

▷ Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on the fixed point

Case 1: terminates

Case 2: loops locally

Case 3: loops globally

Example 5.1: fixed points

Generalizing experience

Fixed point theory

Direct style semantics: existence

Consider the statement

`while $\neg(x = 0)$ do skip`

Functional F' is defined by

$$(F' g) s = \begin{cases} g s & \text{if } s x \neq 0 \\ s & \text{if } s x = 0 \end{cases}$$

Fixed point

$$g_1 s = \begin{cases} \underline{\text{undef}} & \text{if } s x \neq 0 \\ s & \text{if } s x = 0 \end{cases}$$

Not a fixed point: $g_2 s = \underline{\text{undef}}$ for all s

Problem

Overview

Direct style
semantics:
specification

Idea
Semantic
functions so far
DS for While
 S_{ds} for
composition

S_{ds} for condition
 S_{ds} for while

Example 5.1

▷ Problem

Example 5.2

Example 5.3

Requirements on
the fixed point

Case 1: terminates

Case 2: loops
locally

Case 3: loops
globally

Example 5.1: fixed
points

Generalizing
experience

Fixed point theory

Direct style
semantics:
existence

There are functionals

□ with many fixed points

– In example 5.1 every $g' : \mathbf{State} \hookrightarrow \mathbf{State}$ satisfying
 $g' s = s$ if $s x = 0$

□ with no fixed points

– $F_1 g = \begin{cases} g_1 & \text{if } g = g_2 \\ g_2 & \text{otherwise} \end{cases}$

Example 5.2

Overview

Direct style semantics: specification

Idea

Semantic functions so far

DS for While

S_{ds} for composition

S_{ds} for condition

S_{ds} for while

Example 5.1

Problem

▷ Example 5.2

Example 5.3

Requirements on the fixed point

Case 1: terminates

Case 2: loops locally

Case 3: loops globally

Example 5.1: fixed points

Generalizing experience

Fixed point theory

Direct style semantics: existence

Determine functional F for

$$\text{while } \neg(x = 0) \text{ do } x := x - 1$$

Consider

$g_1 s = \underline{\text{undef}}$ for all s

$$g_2 s = \begin{cases} s[x \mapsto 0] & \text{if } s x \geq 0 \\ \underline{\text{undef}} & \text{if } s x < 0 \end{cases}$$

$$g_3 s = \begin{cases} s[x \mapsto 0] & \text{if } s x \geq 0 \\ s & \text{if } s x < 0 \end{cases}$$

$g_4 s = s[x \mapsto 0]$ for all s

$g_5 s = s$ for all s

Example 5.3

Overview

Direct style
semantics:
specification

Idea
Semantic
functions so far
DS for While
 \mathcal{S}_{ds} for
composition

\mathcal{S}_{ds} for condition

\mathcal{S}_{ds} for while

Example 5.1

Problem

Example 5.2

▷ Example 5.3

Requirements on
the fixed point

Case 1: terminates

Case 2: loops
locally

Case 3: loops
globally

Example 5.1: fixed
points

Generalizing
experience

Fixed point theory

Direct style
semantics:
existence

Consider

$\text{while } \neg(x = 1) \text{ do } (y := y * x; x := x - 1)$

Determine F , determine at least 2 fixed points for F

Requirements on the fixed point

Overview

Direct style semantics: specification

Idea
Semantic functions so far
DS for While
 S_{ds} for composition

S_{ds} for condition
 S_{ds} for while

Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on the fixed point
▷ point

Case 1: terminates

Case 2: loops locally

Case 3: loops globally

Example 5.1: fixed points

Generalizing experience

To solve 2 problems with fixed points

- impose requirements, so that there is at most 1
- all functionals from **While** do have such fixed points

To motivate choice of requirements, consider `while b do S`

1. It *terminates*
2. It *loops locally*
 - inside S
3. it *loops globally*
 - on the outer `while`-construct

Fixed point theory

Direct style semantics:

Case 1: terminates

Overview

Direct style
semantics:
specification

Idea
Semantic
functions so far
DS for While
 S_{ds} for
composition

S_{ds} for condition
 S_{ds} for while

Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on
the fixed point

Case 1:

▷ terminates

Case 2: loops
locally

Case 3: loops
globally

Example 5.1: fixed
points

Generalizing
experience

from state s_0 , there are states s_1, \dots, s_n such that

- $\mathcal{B}[[b]] s_i = \begin{cases} \mathbf{tt} & \text{if } i < n \\ \mathbf{ff} & \text{if } i = n \end{cases}$ and
- $\mathcal{S}_{ds}[[S]] s_i = s_{i+1}$ for $i < n$

Now, let g_0 be any fixed point of F , that is
 $F g_0 = g_0$ (chalkboard)

- uniquely defined, i.e. for every fixed point g_0
 - $g_0 s_0 = s_n$

Fixed point theory

Direct style
semantics:

Case 2: loops locally

Overview

Direct style
semantics:
specification

Idea
Semantic
functions so far
DS for While
 S_{ds} for
composition

S_{ds} for condition
 S_{ds} for while

Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on
the fixed point

Case 1: terminates

Case 2: loops

▷ locally

Case 3: loops
globally

Example 5.1: fixed
points

Generalizing
experience

Fixed point theory

Direct style
semantics:
existence

from state s_0 , there are states s_1, \dots, s_n such that

- $\mathcal{B}[[b]] s_i = \mathbf{tt}$ for $i \leq n$ and
- $\mathcal{S}_{ds}[[S]] s_i = \begin{cases} s_{i+1} & \text{for } i < n \\ \underline{\text{undef}} & \text{for } i = n \end{cases}$

Now, let g_0 be any fixed point of F , that is
 $F g_0 = g_0$ (chalkboard)

- uniquely defined, i.e. for every fixed point g_0
 - $g_0 s_0 = \underline{\text{undef}}$

Case 3: loops globally

Overview

Direct style
semantics:
specification

Idea
Semantic
functions so far
DS for While
 S_{ds} for
composition

S_{ds} for condition

S_{ds} for while

Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on
the fixed point

Case 1: terminates

Case 2: loops
locally

Case 3: loops

▷ globally

Example 5.1: fixed
points

Generalizing
experience

Fixed point theory

Direct style
semantics:
existence

There is infinite sequence of states s_1, \dots such that

- $\mathcal{B}[[b]] s_i = \mathbf{tt}$ for all i and
- $\mathcal{S}_{ds}[[S]] s_i = s_{i+1}$ for all i

Now, let g_0 be any fixed point of F , that is
 $F g_0 = g_0$ (chalkboard)

- non-uniquely defined
- cannot obtain any additional requirements for the fixed point

Example 5.1: fixed points

Overview

Direct style semantics: specification

Idea

Semantic functions so far

DS for While

\mathcal{S}_{ds} for composition

\mathcal{S}_{ds} for condition

\mathcal{S}_{ds} for while

Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on the fixed point

Case 1: terminates

Case 2: loops locally

Case 3: loops globally

Example 5.1:

▷ fixed points

Generalizing experience

Fixed point theory

Direct style semantics: existence

Indeed, statement `while $\neg(x = 0)$ do skip` in example 5.1 with functional

$$(F' g) s = \begin{cases} g s & \text{if } s x \neq 0 \\ s & \text{if } s x = 0 \end{cases}$$

and *any* function $g : \text{State} \hookrightarrow \text{State}$ satisfying

$$g s = s \text{ if } s x = 0$$

will be a fixed point of F' .

Intuitively, we are interested in

$$\mathcal{S}_{ds} \llbracket \text{while } \neg(x = 0) \text{ do skip} \rrbracket s_0 = \begin{cases} \underline{\text{undef}} & \text{if } s_0 x \neq 0 \\ s_0 & \text{if } s_0 x = 0 \end{cases}$$

Generalizing experience

Overview

Direct style
semantics:
specification

Idea
Semantic
functions so far
DS for While
 S_{ds} for
composition

S_{ds} for condition

S_{ds} for while

Example 5.1

Problem

Example 5.2

Example 5.3

Requirements on
the fixed point

Case 1: terminates

Case 2: loops
locally

Case 3: loops
globally

Example 5.1: fixed
points

Generalizing
▷ experience

Fixed point theory

Direct style
semantics:
existence

The desired fixed point FIX F should be a partial function $g_o : \mathbf{State} \hookrightarrow \mathbf{State}$ such that

- g_o is a fixed point of F (i.e. $F g_o = g_o$), and
- if g is another f.p. of F , then
 - $g_o s = s'$ implies $g s = s'$

Overview

Direct style
semantics:
specification

Fixed point
▷ theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8

Partially ordered
set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements for
FIX F

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

Fixed point theory

Ordering of semantic functions

Overview

Direct style semantics: specification

Fixed point theory

Ordering of semantic functions

Example 5.6

Exercise 5.8

Partially ordered set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements for $\text{FIX } F$

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style semantics: existence

An equivalence result

Formalize the requirement that

- $\text{FIX } F$ shares its result with all other fixed points

Define

- ordering $g_1 \sqsubseteq g_2$ on partial functions $\text{State} \leftrightarrow \text{State}$

when g_1 shares its result with g_2

- if $g_1 s = s'$ then $g_2 s = s'$

for all choices of s and s'

Example 5.6

Overview

Direct style semantics: specification

Fixed point theory

Ordering of semantic functions

▷ Example 5.6

Exercise 5.8

Partially ordered set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements for FIX F

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style semantics: existence

An equivalence result

g_1, g_2, g_3, g_4 are partial functions $\text{State} \hookrightarrow \text{State}$

- $g_1 s = s$ for all s
- $g_2 s = \begin{cases} s & \text{if } s x \geq 0 \\ \underline{\text{undef}} & \text{otherwise} \end{cases}$
- $g_3 s = \begin{cases} s & \text{if } s x = 0 \\ \underline{\text{undef}} & \text{otherwise} \end{cases}$
- $g_4 s = \begin{cases} s & \text{if } s x \leq 0 \\ \underline{\text{undef}} & \text{otherwise} \end{cases}$

Then $g_1 \sqsubseteq g_1, g_2 \sqsubseteq g_1, g_2 \sqsubseteq g_2$, etc

- Hasse diagram

Exercise 5.8

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

▷ Exercise 5.8

Partially ordered
set \sqsubseteq_D

Example 5.10

Lemma 5.13
Requirements for
FIX F

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

Definition

$$\square \text{ graph}(f) = \{(x, y) \in X \times Y \mid f x = y\}$$

- $(x, y) \in \text{graph}(f)$ and $(x, y') \in \text{graph}(f)$ imply $y = y'$
- $\forall x \in X : \exists y \in Y : (x, y) \in \text{graph}(f)$

\square for partial functions remove the second requirement

An alternative characterization of the ordering \sqsubseteq

\square $g_1 \sqsubseteq g_2$ if and only if $\text{graph}(g_1) \subseteq \text{graph}(g_2)$

Partially ordered set \sqsubseteq_D

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8

Partially
ordered set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements for
FIX F

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

Pair (D, \sqsubseteq_D) that satisfies

1. $d \sqsubseteq_D d$ (reflexivity)
2. $d_1 \sqsubseteq_D d_2$ and $d_2 \sqsubseteq_D d_3$ imply $d_1 \sqsubseteq_D d_3$ (transitivity)
3. $d_1 \sqsubseteq_D d_2$ and $d_2 \sqsubseteq_D d_1$ imply $d_1 = d_2$ (anti-symmetry)

is a partially ordered set.

(often omit D subscript in \sqsubseteq_D)

An element $d \in D$ satisfying

- $d \sqsubseteq d'$ for all $d' \in D$

is called a *least element* of D .

- least element, if exists, is unique; denoted \perp

(**State** \hookrightarrow **State**, \sqsubseteq) is a partially ordered set (Lemma 5.13).

Example 5.10

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8

Partially ordered
set \subseteq_D

▷ Example 5.10

Lemma 5.13

Requirements for
FIX F

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

Let S is a non-empty set, define

$$\mathcal{P}(S) = \{K \mid K \subseteq S\}$$

then $(\mathcal{P}(S), \subseteq)$ is a partially ordered set.

Example case: $S = \{a, b, c\}$

Exercise 5.11 Show that $(\mathcal{P}(S), \supseteq)$ is a partially ordered set, and determine the least element.

Exercise 5.12 Let S be a non-empty set, and define $\mathcal{P}_{\text{fin}}(S) = \{K \mid K \text{ is finite and } K \subseteq S\}$.

- $(\mathcal{P}_{\text{fin}}(S), \subseteq)$ and $(\mathcal{P}_{\text{fin}}(S), \supseteq)$ are partially ordered sets?
- least elements?

Lemma 5.13

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8

Partially ordered
set \sqsubseteq_D

Example 5.10

▷ Lemma 5.13
Requirements for
FIX F

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

$(\mathbf{State} \hookrightarrow \mathbf{State}, \sqsubseteq)$ is a partially ordered set. The partial function $\perp : \mathbf{State} \hookrightarrow \mathbf{State}$ defined by

$$\perp s = \underline{\text{undef}} \text{ for all } s$$

is the least element if $\mathbf{State} \hookrightarrow \mathbf{State}$.

Proof

1. reflexive
2. transitive
3. anti-symmetric

Requirements for $\text{FIX } F$

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8

Partially ordered
set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements

▷ for $\text{FIX } F$

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

More precise requirements using ordering on partial functions

1. $\text{FIX } F$ is a *fixed point* of F
 - i.e. $F(\text{FIX } F) = \text{FIX } F$
2. $\text{FIX } F$ is a *least* fixed point of F
 - i.e. if $F g = g$ then $\text{FIX } F \sqsubseteq g$.

More mathematics

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8

Partially ordered
set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements for
FIX F

More

▷ mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

- Complete partially ordered sets
 - upper bound, least upper bound \sqcup , chain
 - *chain complete* partially ordered sets (ccpo)
 - ▷ ccpo has a least element, given by $\perp = \sqcup \emptyset$
 - complete lattice
 - **State** \hookrightarrow **State** is a ccpo (Lemma 5.25)
- Continuous functions
 - monotone function $f : D \rightarrow D'$ on 2 ccpo's
 - continuous function, preserves least upper bound
 - strict function
- Summarize the existence of least fixed points

Lemma 5.25

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8

Partially ordered
set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements for
FIX F

More mathematics

▷ Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

$(\mathbf{State} \hookrightarrow \mathbf{State}, \sqsubseteq)$ is a ccpo. The least upper bound $\bigsqcup Y$ of a chain Y is given by

$$\text{graph}(\bigsqcup Y) = \bigcup \{\text{graph}(g) \mid g \in Y\}$$

that is,

$$(\bigsqcup Y) s = s' \text{ if and only if } g s = s' \text{ for some } g \in Y$$

Proof

1. show $\bigcup \{\text{graph}(g) \mid g \in Y\}$ is a graph of function in $\mathbf{State} \hookrightarrow \mathbf{State}$
2. prove this function is an upper bound of Y
3. prove it is the least upper bound of Y

Exercise 5.28

Overview

Direct style semantics: specification

Fixed point theory

Ordering of semantic functions

Example 5.6

Exercise 5.8

Partially ordered set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements for FIX F

More mathematics

Lemma 5.25

▷ Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style semantics: existence

An equivalence result

Determine which of the following functionals of $(\mathbf{State} \hookrightarrow \mathbf{State}) \rightarrow (\mathbf{State} \hookrightarrow \mathbf{State})$ are monotone:

□ $F_0 g = g$

□ $F_1 g = \begin{cases} g_1 & \text{if } g = g_2 \\ g_2 & \text{otherwise} \end{cases}$ where $g_1 \neq g_2$

□ $(F' g) s = \begin{cases} g s & \text{if } s x \neq 0 \\ s & \text{if } s x = 0 \end{cases}$

Exercise 5.33

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8

Partially ordered
set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements for
FIX F

More mathematics

Lemma 5.25

Exercise 5.28

▷ Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

Show that the functional F' of Example 5.1 is continuous

$$(F' g) s = \begin{cases} g s & \text{if } s x \neq 0 \\ s & \text{if } s x = 0 \end{cases}$$

Theorem 5.37

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8
Partially ordered
set \sqsubseteq_D

Example 5.10

Lemma 5.13
Requirements for
FIX F

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

▷ Theorem 5.37

Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

Define the required fixed point operator FIX

Let $f : D \rightarrow D$ be a continuous function on the ccpo (D, \sqsubseteq) with least element \perp . Then

$$\text{FIX } f = \bigsqcup \{f^n \perp \mid n \geq 0\}$$

- defines an element of D , and
- this element is the least fixed point of f

Example 5.38

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8

Partially ordered
set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements for
FIX F

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

▷ Example 5.38

Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

Consider the function F' of Example 5.1:

$$(F' g) s = \begin{cases} g s & \text{if } s x \neq 0 \\ s & \text{if } s x = 0 \end{cases}$$

We shall determine its least fix point using the approach of Theorem 5.37.

Exercise 5.40

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8

Partially ordered
set \sqsubseteq_D

Example 5.10

Lemma 5.13

Requirements for
FIX F

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

▷ Exercise 5.40

Summary

Direct style
semantics:
existence

An equivalence
result

Let $f : D \rightarrow D$ be a continuous function on a ccpo (D, \sqsubseteq) and let $d \in D$ satisfy $f d \sqsubseteq d$. Show that $\text{FIX } f \sqsubseteq d$

Summary

Overview

Direct style
semantics:
specification

Fixed point theory

Ordering of
semantic functions

Example 5.6

Exercise 5.8
Partially ordered
set \sqsubseteq_D

Example 5.10

Lemma 5.13
Requirements for
FIX F

More mathematics

Lemma 5.25

Exercise 5.28

Exercise 5.33

Theorem 5.37

Example 5.38

Exercise 5.40

▷ Summary

Direct style
semantics:
existence

An equivalence
result

Fixed point theory

1. We restrict ourselves to *chain complete partially ordered sets* (ccpo)
 - in our case semantic functions ($\mathbf{State} \hookrightarrow \mathbf{State}, \sqsubseteq$)
2. We restrict ourselves to *continuous functions* on ccpo's
 - in our case $F g = \text{cond}(B[[b]], g \circ \mathcal{S}_{ds}[[S]], \text{id})$
3. We show that continuous functions on ccpo's always have *least fixed points*
 - i.e. FIX F always exists

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:

▷ existence

In general

Lemma 5.43
Exercise 5.44
(Essential)

Lemma 5.45
Exercise 5.46
(Essential)

Proposition 5.47

Example 5.48

Summary

Properties of the
semantics

An equivalence
result

Direct style semantics: existence

In general

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

▷ In general

Lemma 5.43
Exercise 5.44
(Essential)

Lemma 5.45
Exercise 5.46
(Essential)

Proposition 5.47

Example 5.48

Summary

Properties of the
semantics

An equivalence
result

Consider again

$$\mathcal{S}_{ds}[\text{while } b \text{ do } S] = \text{FIX } F$$
$$\text{where } F g = \text{cond}(\mathcal{B}[b], g \circ \mathcal{S}_{ds}[S], \text{id})$$

Must show that F is continuous.

Observe

- $F g = F_1(F_2 g)$ where
- $F_1 g = \text{cond}(\mathcal{B}[b], g, \text{id})$ and
- $F_2 g = g \circ \mathcal{S}_{ds}[S]$

Lemma 5.43

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

In general

▷ Lemma 5.43

Exercise 5.44

(Essential)

Lemma 5.45

Exercise 5.46

(Essential)

Proposition 5.47

Example 5.48

Summary

Properties of the
semantics

An equivalence
result

Let $g_0 : \mathbf{State} \hookrightarrow \mathbf{State}$, $p : \mathbf{State} \rightarrow \mathbf{T}$, and define

$$F g = \text{cond}(p, g, g_0)$$

then F is continuous.

Proof

1. F is monotone

$$\square \quad g_1 \sqsubseteq g_2 \text{ implies } F g_1 \sqsubseteq F g_2$$

2. F is continuous: let Y is a non-empty chain

$$\square \quad \text{must show that } F(\bigsqcup Y) \sqsubseteq \bigsqcup \{F g \mid g \in Y\}$$

$$\square \quad \text{i.e. } \text{graph}(F(\bigsqcup Y)) \subseteq \bigcup \{\text{graph}(F g) \mid g \in Y\}$$

$$- \quad \text{see slide 35 for } \text{graph}(\bigsqcup Y) = \bigcup \{\text{graph}(g) \mid g \in Y\}$$

Exercise 5.44 (Essential)

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

In general

Lemma 5.43

Exercise 5.44

▷ (Essential)

Lemma 5.45

Exercise 5.46

(Essential)

Proposition 5.47

Example 5.48

Summary

Properties of the
semantics

An equivalence
result

Prove (in the setting of Lemma 5.43) F defined by

$$\square \quad F g = \text{cond}(p, g_0, g)$$

is continuous.

Lemma 5.45

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

In general

Lemma 5.43
Exercise 5.44
(Essential)

▷ Lemma 5.45
Exercise 5.46
(Essential)

Proposition 5.47
Example 5.48

Summary
Properties of the
semantics

An equivalence
result

Let $g_0 : \mathbf{State} \hookrightarrow \mathbf{State}$, and define

$$F g = g \circ g_0$$

then F is continuous.

Proof:

- show that F is monotone, use
 - $\text{graph}(g_1) \subseteq \text{graph}(g_2) \implies$
 - $\text{graph}(g_0) \diamond \text{graph}(g_1) \subseteq \text{graph}(g_0) \diamond \text{graph}(g_2)$
- show that F is continuous
 - i.e. $\text{graph}(F(\bigsqcup Y)) = \dots = \text{graph}(\bigsqcup \{F g \mid g \in Y\})$
 - Lemma 5.25 is used twice

Exercise 5.46 (Essential)

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

In general

Lemma 5.43
Exercise 5.44
(Essential)

Lemma 5.45
Exercise 5.46
▷ (Essential)

Proposition 5.47

Example 5.48

Summary

Properties of the
semantics

An equivalence
result

Prove that (in the setting of Lemma 5.45) F defined by

$$\square \quad F g = g_0 \circ g$$

is continuous.

Proposition 5.47

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

In general

Lemma 5.43
Exercise 5.44
(Essential)

Lemma 5.45
Exercise 5.46
(Essential)

Proposition
▷ 5.47

Example 5.48

Summary

Properties of the
semantics

An equivalence
result

The semantic equations of Table 5.1 define a total function \mathcal{S}_{ds} in $\mathbf{Stm} \rightarrow (\mathbf{State} \leftrightarrow \mathbf{State})$.

Proof

By structural induction on statement S

1. $x := a$
2. `skip`
3. $S_1; S_2$
4. `if b then S_1 else S_2`
5. `while b do S`

Example 5.48

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

In general

Lemma 5.43
Exercise 5.44
(Essential)

Lemma 5.45
Exercise 5.46
(Essential)

Proposition 5.47

▷ Example 5.48

Summary

Properties of the
semantics

An equivalence
result

Consider the denotational semantics of the factorial statement

$$\mathcal{S}_{ds} \llbracket y := 1; \text{while } \neg(x = 1) \text{ do } (y := y * x; x := x - 1) \rrbracket$$

Summary

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

In general

Lemma 5.43
Exercise 5.44
(Essential)

Lemma 5.45
Exercise 5.46
(Essential)

Proposition 5.47

Example 5.48

▷ Summary

Properties of the
semantics

An equivalence
result

The well-definedness of \mathcal{S}_{ds} relies on the following results

1. The set $\mathbf{State} \hookrightarrow \mathbf{State}$ equipped with \sqsubseteq is a ccpo
2. Certain functions $\Psi : (\mathbf{State} \hookrightarrow \mathbf{State}) \rightarrow (\mathbf{State} \hookrightarrow \mathbf{State})$ are continuous
3. In the definition of \mathcal{S}_{ds} we only apply the fixed point operation to continuous functions

Properties of the semantics

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

In general

Lemma 5.43
Exercise 5.44
(Essential)

Lemma 5.45
Exercise 5.46
(Essential)

Proposition 5.47

Example 5.48

Summary

▷ Properties of
the semantics

An equivalence
result

S_1 and S_2 are semantically equivalent if and only if

$$\mathcal{S}_{ds}[[S_1]] = \mathcal{S}_{ds}[[S_2]]$$

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

▷ An equivalence
result

Theorem 5.55

Lemma 5.56

Lemma 5.57

Proof summary for
While

An equivalence result

Theorem 5.55

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

An equivalence
result

▷ Theorem 5.55

Lemma 5.56

Lemma 5.57

Proof summary for
While

For every statement S of **While**, we have $\mathcal{S}_{sos}[[S]] = \mathcal{S}_{ds}[[S]]$.

Proof:

- $\mathcal{S}_{sos}[[S]] \subseteq \mathcal{S}_{ds}[[S]]$ and (Lemma 5.56)
- $\mathcal{S}_{ds}[[S]] \subseteq \mathcal{S}_{sos}[[S]]$ (Lemma 5.57)

Lemma 5.56

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

An equivalence
result

Theorem 5.55

▷ Lemma 5.56

Lemma 5.57
Proof summary for
While

For every statement S of **While**, we have $\mathcal{S}_{sos}[[S]] \subseteq \mathcal{S}_{ds}[[S]]$.

Proof: It is sufficient to prove that for all s and s' (*)

$$\langle S, s \rangle \Rightarrow^* s' \text{ implies } \mathcal{S}_{ds}[[S]] s = s'$$

To do so, we shall need to establish (**)

$$\begin{aligned} \langle S, s \rangle \Rightarrow s' & \text{ implies } \mathcal{S}_{ds}[[S]] s = s' \\ \langle S, s \rangle \Rightarrow \langle S', s' \rangle & \text{ implies } \mathcal{S}_{ds}[[S]] s = \mathcal{S}_{ds}[[S']] s' \end{aligned}$$

- Then by induction on the length k of of the derivation sequence $(**) \implies (*)$
- Prove $(**)$ by the induction on the shape of the derivation tree for $\langle S, s \rangle \Rightarrow s'$ and $\langle S, s \rangle \Rightarrow \langle S', s' \rangle$

Lemma 5.57

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

An equivalence
result

Theorem 5.55

Lemma 5.56

▷ Lemma 5.57
Proof summary for
While

For every statement S of **While**, we have $\mathcal{S}_{ds}[[S]] \sqsubseteq \mathcal{S}_{sos}[[S]]$

Proof: By structural induction on the statement S .

- case $S_1; S_2$: use \circ is monotone + exercise 2.21 result
 - ex. 2.21: if $\langle S_1, s \rangle \Rightarrow^* \langle S_2, s' \rangle$ then $\langle S_1; S_2, s \rangle \Rightarrow^* \langle S_2, s' \rangle$
- case if b then S_1 else S_2 : use cond is monotone
- case while b do S
 - prove $F(\mathcal{S}_{sos}[\text{while } b \text{ do } S]) \sqsubseteq \mathcal{S}_{sos}[\text{while } b \text{ do } S]$
 - ex. 5.40: let $f : D \rightarrow D$ be a continuous function on a ccpo (D, \sqsubseteq) and let $d \in D$ satisfy $f d \sqsubseteq d$. Show that $\text{FIX } f \sqsubseteq d$

Proof summary for While

Overview

Direct style
semantics:
specification

Fixed point theory

Direct style
semantics:
existence

An equivalence
result

Theorem 5.55

Lemma 5.56

Lemma 5.57

Proof summary
▷ for While

Equivalence of Operational and Denotational Semantics

1. Prove that $\mathcal{S}_{sos} \llbracket S \rrbracket \sqsubseteq \mathcal{S}_{ds} \llbracket S \rrbracket$ to show that
 - (a) if a statement is executed *one step* in SOS and
 - i. does not terminate, then this does not change the meaning of DS
 - ii. does terminate, then the same result is obtained in DS
 - (b) and using *induction on the length of derivation sequences*.
2. Prove that $\mathcal{S}_{ds} \llbracket S \rrbracket \sqsubseteq \mathcal{S}_{sos} \llbracket S \rrbracket$
 - structural induction on statement S